THE COMBINATORICS AND GEOMETRY OF THE ORBITS OF THE SYMPLECTIC GROUP ON FLAGS IN COMPLEX AFFINE SPACE

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Let $\mathcal{F}l\mathbb{C}^{2n} = B_-\backslash GL_{2n}\mathbb{C}$ be the manifold of flags in $\mathbb{C}^{2n}$. $\mathcal{F}l\mathbb{C}^{2n}$ has a natural action of Sp_n by right multiplication. In this thesis we will describe the orbits of Sp_n on $\mathcal{F}l\mathbb{C}^{2n}$. We begin by giving background material in chapter 2 on the combinatorics of S_n, the flag manifold, and Gröbner bases. In chapter 3 we describe the orbits of $B_- \times Sp_n$ on full rank $2n \times 2n$ matrices (equivalent to the orbits of Sp_n on $\mathcal{F}l\mathbb{C}^{2n}$) by mapping those orbits to orbits of $B_- \times B_+$ via $M \mapsto MJM^T$ using [RS90] and then applying the tools available to understand those orbits (see [Ful92]). We recall that the orbits of $B_- \times Sp_n$ on full rank matrices correspond to fixed-point-free involutions and we explore the combinatorics of the poset of fixed point free involutions to gain insight into the corresponding poset of orbit closures. We also give a Gröbner degeneration of each orbit closure to a union of matrix Schubert varieties. In the chapter 4 we develop understanding of unions of matrix Schubert varieties by finding their equations. In chapter 5 we give the partial results that we have achieved in finding the defining equations for the orbit closures of the orbits of $B_- \times Sp_n$.

BIOGRAPHICAL SKETCH

Anna Bertiger is a mathematician who enjoys cooking, baking, running, yoga and riding horses. She grew up in the Los Angeles area and first fell in love with mathematics as an undergraduate at the University of Chicago, where she received a bachelor's degree in 2006. She obtained a master's degree from the University of California at San Diego in 2008 before continuing to Cornell University.

To Mrs. Reeves, the first in a long line of excellent math teachers from whom I have been lucky enough to learn.

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I'm grateful to everyone who has worked at Gimme! Coffee on Cayuga Street in the last few years, especially Steve Zimmerman and Sam Queen. A large portion of the research and writing in this thesis was done at Gimme and they gave me a warm, supportive place to work and sustained me with coffee, scones and friendship.

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CHAPTER 1

BACKGROUND

One general reference for this chapter is [MS05], especially chapters 14-17.

1.1 Combinatorics of S_n

The **symmetric group** S_n is the group of permutations on the letters $\{1, \ldots, n\}$. Typically in what follows we will write the elements of a permutation π in **one line** notation, as $\pi(1) \ldots \pi(n)$ and write s_i for the **simple transposition** that switches i and $i + 1$ and fixes all other elements. A **reduced word** for a permutation π will be a list of simple transpositions $s_{i_1} \ldots s_{i_l}$ that when multiplied together give π with l as small as possible. This smallest possible l will be the **length** of the permuation π, denoted $l(\pi)$. Note that reduced words are not unique for a given permutation; for example the permutation 321 has reduced words $s_1 s_2 s_1$ and $s_2 s_1 s_2$. Another means of presenting a permutation that we will use frequently is the $n \times n$ permutation matrix M_π in which $(M_\pi)_{ij} = 1$ if $\pi(i) = j$ and 0 otherwise. The **Rothe diagram** of a permutation is found by looking at the permutation matrix and crossing out all of the cells weakly below, and the cells weakly to the right of, each cell containing a 1. The remaining empty boxes form the Rothe diagram. The **essential boxes** [Ful92] of a permutation are those boxes in the Rothe diagram that do not have any boxes of the diagram immediately south or east of them. The Rothe diagram for 2143 is given in figure 1.1 and the Rothe diagram for 15432 is given in figure 1.2. In both cases the essential boxes are marked with red dots.

Theorem 1.1.1 *The length of the permutation is given by the number of boxes in the*

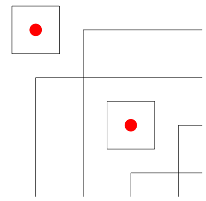

Figure 1.1: The Rothe diagram and essential set of 2143

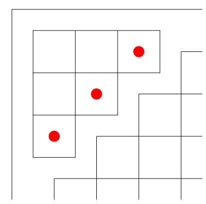

Figure 1.2: The Rothe diagram and essential set of 15432

Rothe diagram.

The **rank matrix** of π, $r(\pi)$, has entries $r_{ij}(\pi) = \#\{k \leq i : \pi(k) \leq j\}$. For example, the rank matrix of 13425 is given by:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

We will impose the **weak (Bruhat) order** as a partial order on elements of S_n, where π covers ρ if $\pi = \rho s_i$ and $l(\pi) = l(\rho) + 1$. For example, 321 covers 231 and $321 \geq 123$ but 231 and 132 have no relationship. We will also impose the **(strong) Bruhat order** on elements of S_n, where π covers ρ if $\pi = \rho t_{ij}$ and $l(\pi) = l(\rho) + 1$, where t_{ij} is the transposition that switches i and j and fixes all other elements. In strong Bruhat order, 231 does cover 132. Note that the strong Bruhat order is also given by entrywise comparison of the rank matrices of the two permutations, that is $\pi \geq \rho$ if and only if $r_{ij}(\pi) \geq r_{ij}(\rho)$ for all i and j.

1.2 Flag Manifolds

The **flag manifold** $\mathcal{F}l\mathbb{C}^n$ is composed of nested chains of vector subspaces $\{V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n\}$ of $\mathbb{C}^n$ where each subspace V_i is of dimension i. We can also think of this as $B \backslash GL_n$, where GL_n is the invertible $n \times n$ matrices and $B = B_-$ is the subgroup of lower triangular $n \times n$ matrices. It forms a manifold when endowed with the quotient topology. A **Schubert variety** X_π is $B\backslash\overline{BM_\pi B_+}$, where B_+ is the subgroup of upper triangular invertible matrices. The flag manifold is stratified by Schubert varieties. A **matrix Schubert variety** $\overline{X_\pi}$ is the closure of $B_- M_\pi B_+$ in the space of all $n \times n$ matrices.

Theorem 1.2.1 ([Ful92]) *Matrix Schubert varieties have radical ideal $I(\overline{X_\pi}) = I_\pi$ given by determinants representing conditions given in the rank matrix $r(\pi)$, that is, the $(r(\pi)_{i,j} + 1) \times (r(\pi)_{i,j} + 1)$ determinants of each northwest $i \times j$ submatrix of a matrix of variables. In fact, it is sufficient to impose only those rank conditions $r_{ij}(\pi)$ such that (i, j) is an essential box for π.*

We will call these determinants or the analogous determinants for any ideal generated by northwest rank conditions the **Fulton generators**. For example, the Fulton generator of $I(\overline{X_{1,324}})$ is $-m_{1,2}m_{2,1} + m_{1,1}m_{2,2}$ and the Fulton generators for $I(\overline{X_{312}})$ are $m_{1,1}$ and $m_{1,2}$.

1.3 Gröbner Bases

Fix a total ordering on the monomials of a polynomial ring, perhaps by putting incomensurable weightings on the variables and comparing the total weight of the variables (with multiplicity) appearing in each monomial. If $1 < m$ for all monomials m and if $m < n$ then $pm < pn$ for all monomials p we will call such an ordering of the monomials a **term order**. The largest monomial appearing in any polynomial f is the **initial term** of f, denoted init f. Define the **initial ideal** of an ideal I to be init $I := \langle \text{init } f : f \in I \rangle$. A subset $f_1, \ldots, f_r$ of I is a **Gröbner basis** for I if init $I = \langle \text{init } f_1, \ldots, \text{init } f_r \rangle$ and this implies that a Gröbner basis for I is also a generating set for I. If I is the defining ideal for a scheme, the scheme corresponding to init I is geometrically related to the one corresponding to I, in that the initial scheme can be found from the original scheme by deforming along a flat family.

A **simplicial complex** Δ on a set X is a subset of the power set $\mathcal{P}(X)$ such that if $Y \in \Delta$ and $Y' \subseteq Y$ then $Y' \in \Delta$. The **Stanley-Reisner ideal** I_Δ of a simplicial complex is the ideal generated by monomials formed by taking products of the variables corresponding to the non-faces of the simplicial complex.

The **antidiagonal** of a matrix is the diagonal series of cells in the matrix running from the most northeast to the most southwest cell. The **antidiagonal term** (or **antidiagonal**) of a determinant is the product of the entries in the antidiagonal. For example, the antidiagonal of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the cells occupied by b and c, and correspondingly, in the determinant $ad - bc$ the antidiagonal term is bc. Typically these will be denoted A or B in what follows, with the notions of sets of cells and monomials interchanged freely. Term orders that select antidiagonal terms from a determinant, called **antidiagonal term orders** have proven espeically useful in understanding ideals of matrix Schubert varieties. There are several possible implementations of an antidiagonal term order on an $n \times n$ matrix of variables, for example rastering across the matrix right to left and top to bottom.

Theorem 1.3.1 ([KM05]) *The Fulton generators for the matrix Schubert variety I_π form a Gröbner basis for I_π under any antidiagonal term order. Further, the corresponding initial ideal initI_π is the Stanley-Reisner ideal of a shellable simplicial complex known as the "pipe dream complex."*

Theorem 1.3.2 ([Knu]) *If $\{I_i : i \in S\}$ are ideals generated by northwest rank conditions, then init $\left(\cap_{i \in S} I_i\right) = \cap_{i \in S}(\text{init } I_i)$ for any antidiagonal term order.*

Note that the pipe dream complex is shellable, hence the corresponding ideals are Cohen-Macaulay (see [KM05]).

CHAPTER 2

ORBITS OF SP_N ON $\mathcal{F}L\mathbb{C}^{2N}$ VIA ORBIT DEGENERATION

Let J be the $2n \times 2n$ block diagonal matrix with diagonal blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The **symplectic group** Sp_n is the subgroup of invertible $2n \times 2n$ matrices with complex entries that preserve the symplectic form J, i.e. Sp_n is $\{M \in GL_{2n}(\mathbb{C}) : MJM^T = J\}$.

2.1 Orbits of the Action of Sp_n on $\mathcal{F}l\mathbb{C}^{2n}$

A **fixed-point-free involution** is an element $\iota \in S_{2n}$ such that $\iota^2 =$ identity and $\iota(i) \neq i$ for $1 \leq i \leq 2n$.

Theorem 2.1.1 ([RS90]) *Sp_n orbits on $\mathcal{F}l\mathbb{C}^n$ correspond to fixed-point-free involutions.*

Proof See [RS93] example 5.1(4) and [RS90] section 10. We map from

$$\{\text{orbits of } B_{-}{\times}Sp_n \text{ on } 2n{\times}2n \text{ full rank matrices}\} \to \{\text{orbits of } B_{-} \text{ acting by } b{\cdot}M = bMb^T\}$$

via the map on matrices $M \mapsto MJM^T$. □

We shall denote the orbit corresponding to the fixed-point-free involution ι Y_ι.

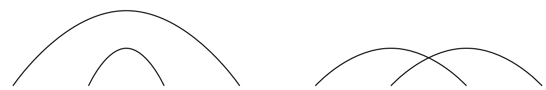

Figure 2.1: The wiring diagram for 43217856.

2.2 Combinatorics of the Poset of Fixed-Point-Free Involutions

Begin with a $1 \times 2n$ array of dots which we will call **outlets**. A **wiring diagram** for a fixed-point-free involution is the figure formed by connecting the i^{th} and $\iota(i)^{th}$ outlets with an arc, or wire, run over the array of dots.

Notice that the fixed-point-free involutions of $1, \ldots, 2n$ inherit a partial order from the (weak or strong) Bruhat order on S_{2n}. The covering relations are now given by *conjugating by* a transposition (simple transposition in the weak case) resulting in an increase in length by 2.

Proposition 2.2.1 ([Inc04]) *The covering relations in the weak Bruhat order between fixed-point-free involutions (thought of as wiring diagrams) can be described by switching the plugs in two adjacent outlets such that the length of the new involution goes up by* 2*. Similarly, the covering relation from the strong Bruhat order on fixed-point-free involutions is given by switching the plugs in any pair of outlets such that the length goes up by* 2*.*

Proof The covering relation inherited from the weak Bruhat order is two permutations that differ in length by two and differ by two adjacent transpositions. We require that we still have a fixed-point-free involution when we are done applying two adjacent transpositions on the right (acting on entries). To accomplish this, when we switch one pair of adjacent entries i and $i+1$ we must

Figure 2.2: The wiring diagrams for 341265 and 214365.

also switch the entries that are given by the numbers in these entries $\pi(i)$ and $pi(i+1)$. The above proof also applies for the strong Bruhat order case, where simple transpositions are replaced by arbitrary transpositions. □

For example, if we switch the 1 and the 4 in 214365 we must also switch the 2 and 3. This is the same as switching the plugs corresponding to the ends of the wire connecting 1 and 2 and the wire connecting 3 and 4. In this order 341265 is covered by 214365. Both of these permutations are displayed in figure 2.2.

Note that in the strong Bruhat order we can switch non-adjacent plugs, provided that we only change the length of the permutation by 2.

Let ι be an involution of $1, \ldots, 2k$ and ι' be an involution of $1, \ldots, 2m$. We shall define the involution $\iota \oplus \iota'$ of $1, \ldots 2k + 2m$ by

$$(\iota \oplus \iota')(i) = \begin{cases} \iota(i) & i \in \{1, \ldots, 2k\} \\ \iota'(i-2k) + 2k & i \in \{2k+1, \ldots, 2k+2m\} \end{cases}$$

It will also be useful to define a special involution $\overline{J}_n$ to be the fixed-point-free involution in S_{2n} that switches $2i-1$ and $2i$ for $1 \leq i \leq n$. This is the combinatorial shadow of the symplectic form J. We will say that ι is a **countryside involution** if ι is of the form $\overline{J}_k \oplus 3412 \oplus \overline{J}_{n-k-2}$ and that ι is a **rainbow permutation** if ι is of the form $\overline{J}_k \oplus 4321 \oplus \overline{J}_{n-k-2}$.

We will say that one involution ι **contains** another involution ι' if by removing enough arcs from the wiring diagram of ι we are left with the wiring diagram for ι'. For example, 43217856 contains both a rainbow involution and a

Figure 2.3: 21563487 is a countryside involution.

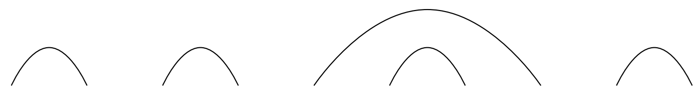

Figure 2.4: 21438765109 is a rainbow involution.

countryside involution as shown in figure 2.5.

Notice that this notion of containment matches the notion of containment from pattern avoidance in permutations defined by removing sufficiently many rows and columns from a permutation matrix. In the case of fixed-point-free involutions we require that row and column removal is done such that the permutation continues to be a fixed-point-free involution, i.e. that the same rows as columns are removed.

Lemma 2.2.2 *The fixed-point-free involutions with only one cover in the order inherited from Bruhat order on S_{2n} are the countryside involutions and the rainbow involutions.*

Proof That these permutations have only one cover is clear: for a countryside involution the only cover is $\overline{J}_n$, and for a rainbow involution the only cover is

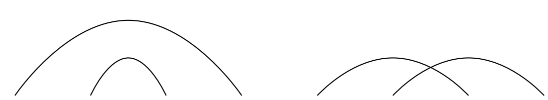

Figure 2.5: 43217856 contains both a rainbow involution and a countryside involution.

the corresponding two hills.

Take any other kind of permutation. It has more than one pair of plugs that can be reversed to get different permutations of length two smaller. □

Notice that we can write an involution by writing a history of plug switches. This is the analog of a reduced word in S_n, and is similarly not unique. In fact, switching plugs i and $i+1$ corresponds conjugating by the simple reflection s_i.

Lemma 2.2.3 *The permutation length of a fixed-point-free involution ι of $1, \ldots, 2n$ is given by $n + 2c + 4r$, where c is the number of countryside involutions contained in ι and r is the number of rainbow involutions contained in r.*

Proof By induction on the length of the permutation. For the base case examine the fixed-point-free involution $\overline{J}_n$ which has reduced word given by $s_1 s_3 \cdots s_{2n-1}$ and length $n = n + 2c + 4r$, since c and r are both 0. Now assume that this is true for all fixed-point-free involutions of length at most l for a fixed $l \geq n$. Fix a fixed-point-free involution ι of length $l + 1$ and examine its wiring diagram. There are two adjacent plugs whose wires can be interchanged to produce a shorter permutation ι'. Pick the rightmost pair of adjacent plugs at locations i and $i + 1$ such that $\iota(i) \neq i + 1$ and $\iota(i) > \iota(i + 1)$. Conjugating ι by s_i, i.e. switching the i and $i + 1$ plugs, produces an involution of length l, and hence one to which our inductive hypothesis applies. This switch also changes the two wires involved in one of two ways: it turns them from a countryside involution to the involution $\overline{J}_2$ or from a rainbow involution to a countryside involution. Each of these changes reduces the equation $n + 2c + 4r$ by two. □

The **symplectic essential boxes** for a fixed-point-free involution ι are the essential boxes (using the same definition as for theorem 1.2.1) in the **symplectic**

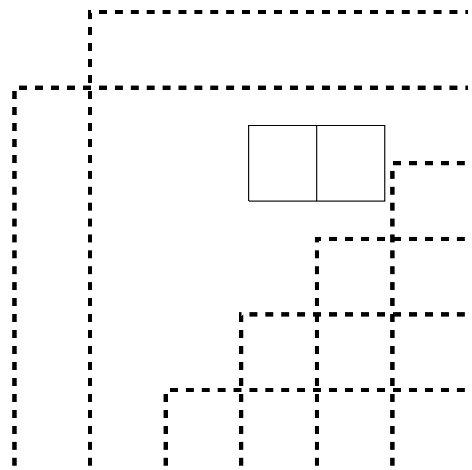

Figure 2.6: The symplectic diagram for 216543.

diagram for ι formed by taking the usual Rothe diagram for ι and intersecting it with the strict upper triangle. The symplectic essential box for 216543 is at $(2, 5)$; the boxes in the symplectic diagram are shown in figure 2.6.

The **basic elements** of a partially ordered set are the elements from which all other elements can be found by taking the unique least upper bound of smaller basic elements in the poset (see [Knu]). These elements give us key parts of the structure of the poset. Further, if this poset comes from a stratification, each subset can be found by intersecting the basic subsets containing it. For example, the basic elements in $(S_n)^{op}$ are those permutations with only one essential box. The Schubert varieties corresponding to these elements of S_n are the basic elements in the set of Schubert varieties ordered by containment. If the subsets in the poset are compatibly split (as they are in this case) then finding the equations for the basic elements is sufficient to find equations for all elements as intersecting varieties corresponds to summing their ideals, i.e. concatenating their list of generators.

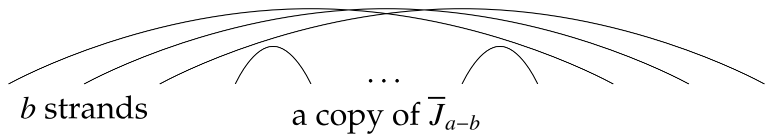

Figure 2.7: $\iota_e(a, b)$

We will now introduce a set that contains the basic elements of the partially ordered set of fixed-point-free involutions, Define the subset $\mathcal{A}$ of the fixed-point-free involutions to be the union of the sets $\mathcal{A}_e$ and $\mathcal{A}_o$, where $\mathcal{A}_e$ is the set of all fixed-point-free involutions with exactly one symplectically essential box which corresponds to an even rank condition and $\mathcal{A}_o$ is the set of all fixed-point-free involutions with exactly two symplectically essential boxes in rows p and $p + 1$, where the rank condition in the symplectically essential box in row p is $2r$ and in the symplectically essential box in row $p + 1$ is $2r + 1$.

Note that we can find fixed-point-free involutions with one even symplectically essential box in any location in which it is possible to have an even rank condition. To obtain a symplectically essential rank condition $2r$ at (i, j) where $i > 2r$ and $j > 2r$ and no other symplectically essential boxes, we need two cases depending on whether $i - j$ is even or odd. If $i - j$ is even, the permutation $\overline{J}_r \oplus \iota_e(i-2r, j-2r) \oplus \overline{J}_{n-r-(i+j)/2}$ has exactly one symplectically essential box at (i, j) with rank condition $2r$. $\iota_e(a, b)$ is shown in figure 2.7.

If $i - j$ is odd, the permutation $\overline{J}_r \oplus \iota'_e(i - 2r, j - 2r) \oplus \overline{J}_{n-r-(i+j+1)/2}$ has exactly one symplectically essential box at (i, j) with rank condition $2r$. $\iota'_e(a, b)$ is shown in figure 2.8. For example, 216543 has one symplectically essential box at $(3, 5)$ which has rank condition 2 associated to it.

The odd symplectic essential rank conditions are a little bit more delicate.

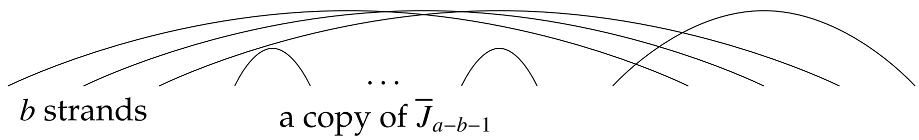

Figure 2.8: $\iota'_e(a,b)$

We begin with a lema restricting the locations of odd symplectic essential rank conditions.

Lemma 2.2.4 *If a fixed-point-free involution has a box in its symplectic diagram at location* (i, j) *with an odd rank condition* $2k+1$ *then the rank condition at box* $(i-1, i)$ *must be at most* $2k$*.*

Proof The rank condition in cell $(i-1, i-1)$ must be even (it is an antisymmetric matrix) and at most $2k+1$, so therefore must be at most $2k$. If the rank condition associated to cell $(i-1, i)$ is larger than $2k$ there must be a 1 in column i in the permutation matrix above row $i-1$. If that is the case then there must be a 1 in the permutation matrix in row i to the left of column $i-1$, contradicting that the box (i, j) was in the diagram of the fixed-point-free involution. □

Note that lemma 2.2.4 implies that we cannot place a symplectically essential box with an odd rank condition on the immediate super-diagonal. However, we can put a symplectically essential box with an odd rank condition anywhere else with an even essential box in the previous row on the immediate superdiagonal and no other symplectically essential boxes. To obtain a symplectically essential rank condition of $(2r+1)$ in box (i, j) we again need to do a case analysis on $i-j$. If $i-j$ is even use the fixed-point-free involution $\overline{J}_r \oplus \iota_o(i-2r, j-2r) \oplus \overline{J}_{r-(i+j)/2}$, where $\iota_o(a,b)$ is shown in figure 2.9. If $i-j$ is odd use the fixed-point-free involution

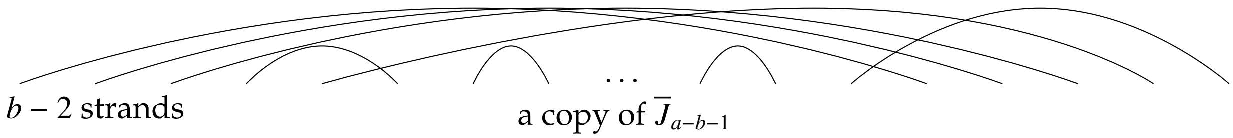

Figure 2.9: $\iota_o(a,b)$

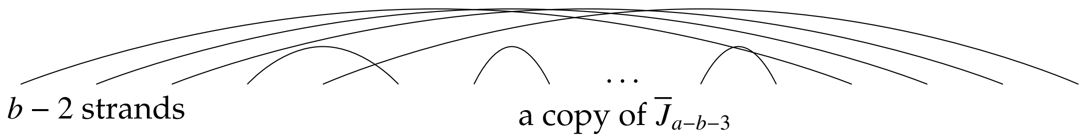

Figure 2.10: $\iota'_o(a,b)$

$\overline{J}_r \oplus \iota'_o(i-2r, j-2r) \oplus \overline{J}_{r-(i+j-1)/2}$, where $\iota'_o(a,b)$ is shown in figure 2.10. For example, 21573846 has one even sympletically essential box at $(3,4)$ with associated rank condition 2 and one odd sympletically essential box at $(4,6)$ with associated rank condition 3.

Proposition 2.2.5 *The set $\mathcal{A}$ contains the basic elements of the poset of fixed-point-free involutions under the (strong) Bruhat order.*

Proof We will show each element is the greatest lower bound of some elements of $\mathcal{A}$. Fix a fixed-point-free involution ι.

For each symplectically essential box at (i, j) in the diagram of ι with even rank condition r, $\iota \leq \iota'$ where ι' is the inolution in $\mathcal{A}_e$ with exactly one symplectically essential box at (i, j) of even rank condition r.

For each symplectically essential box at (i, j) in the diagram of ι with odd rank condition r, fix ι' where ι' is the involution in $\mathcal{A}_o$ with exactly two symplectically essential rank conditions, rank condition r at (i, j) and rank condition

$r-1$ at $(i-1, i)$.

By lemma 2.2.4, $\iota \leq \iota'$ for all of the ι' chosen above. Since for each of ι's symplectically essential rank conditions we have provided an ι' such that $\iota \leq \iota'$, $\iota \leq \mathrm{glb}\{\iota'\}$. Further, as we have imposed no extra rank conditions not met by ι, $\iota \geq \mathrm{glb}\{\iota'\}$. $\square$

2.3 Orbit Degeneration

In this section we will show that the Y_ι degenerate to unions of matrix Schubert varieties. In this case we use a variant of the anti-diagonal term order in which we allow ties in the ordering of monomial terms such that the northwest-most determinant in a sum of determinants will be picked as the initial form for the sum. This can be accomplished for MJM^T by weighting the columns of M: assign weight $t^{\lceil j/2 \rceil - 1}$ to the variable m_{ij}. Then, we shall find initial forms by taking the limit as $t \to 0$.

A **pair permutation** for a fixed-point-free involution ι is a permutation found by this process:

- Write the wiring diagram for ι across the top of an array
- Write the wiring diagram for $\overline{J}_n$ along the left side of an array
- Connect the half loops formed by the wires into circles with more wires, so as the minimize the number of potential wire crossings
- Read off of the newly added wires a permutation by reading this as the wiring diagram for a permutation

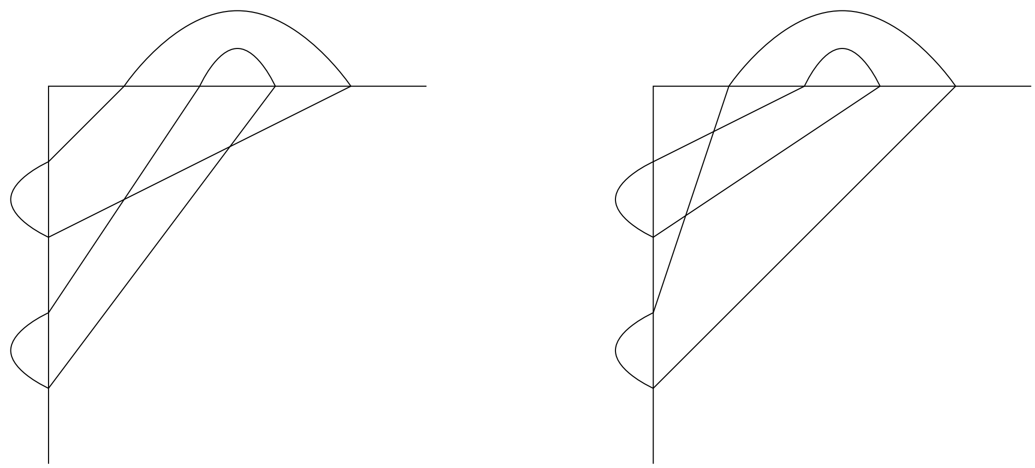

Figure 2.11: The pair permutations for 4321.

The pair permutations for 4321 are 1423 and 2314. The diagrams are shown in figure 2.11.

Theorem 2.3.1 *Y_ι degenerates to the union of Schubert varieties over the pair permutations corresponding to ι.*

The equations for these orbit degenerations are given in theorem 3.2.1. In order to prove 2.3.1 we will first need some lemmas:

Lemma 2.3.2 *If w is a pair permutation for ι, then $X^w_\circ$ and $(B_-\backslash Y^\circ_\iota)$ intersect transversely in the reduced point $B_-\backslash B_-w$.*

Proof We need to show that $T_{B\backslash w}X^w \cap T_{B\backslash w}Y_\iota = \{0\} \subseteq T_{B\backslash w}B$. To show this we shall show that $(\mathfrak{b}_-w^{-1} + w^{-1}\mathfrak{b}_-) \cap (\mathfrak{b}_-w^{-1} + w^{-1}\mathfrak{sp}_n) = \mathfrak{b}_-w^{-1}$ by showing the equivalent equation $(w\mathfrak{b}_-w^{-1}+\mathfrak{b}_-)\cap(w\mathfrak{b}_-w^{-1}+\mathfrak{sp}_n) = w\mathfrak{b}_-w^{-1}$, i.e. $\mathfrak{b}_-\cap\mathfrak{sp}_n \subseteq w\mathfrak{b}_-w^{-1}$.

Notice first that

$$\mathfrak{sp}_n = \{[A_{ij}]_{1\le i,j\le n} : A_{ij} \text{ is a } 2\times 2 \text{ matrix s.t. } A_{ji} = JA_{ij}^T J\}$$

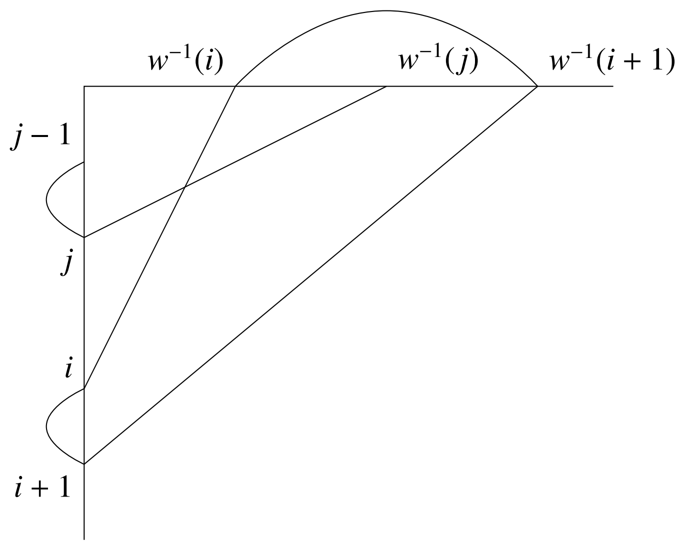

Figure 2.12: The proof of lemma 2.3.2 when i is odd and j is even.

where J is the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Fix an entry (i, j) of a matrix $M \in \mathfrak{b}_- \cap \mathfrak{sp}_n$. We shall show that if it is not in $w\mathfrak{b}_- w^{-1}$ then it is equal to 0. The only case to check is $i > j$ and $w^{-1}(i) < w^{-1}(j)$. In this case we must show that this entry is equal to some entry that is 0 in $w\mathfrak{b}_- w^{-1}$, i.e. an entry (a, b) such that $a < b$ and $w(a) < w(b)$. There are four cases to check.

Case i odd and j is even: In this case the equations for $M \in \mathfrak{sp}_n$ require that $M_{i,j} = M_{j-1,i+1}$. The conditions of being a pair permutation, along with the assumption that $i > j$ and $w^{-1}(i) < w^{-1}(j)$ then require that $w^{-1}(j-1) < w^{-1}(j)$ and $w^{-1}(j) < w^{-1}(i+1)$. This is illustrated in figure 2.3.

Case i is even and j is odd: This case cannot actually occur. The equations for $M \in \mathfrak{sp}_n$ require $M_{i,j} = M_{j+1,i-1}$. If $i = j+1$ the definition of a pair permutation prohibits $w^{-1}(i) < w^{-1}(j)$. If $i \neq j+1$, the minimum length condition of the pair permutation again prohibits this possibility because $w^{-1}(j+1) > w^{-1}(j) > w^{-1}(i) >$

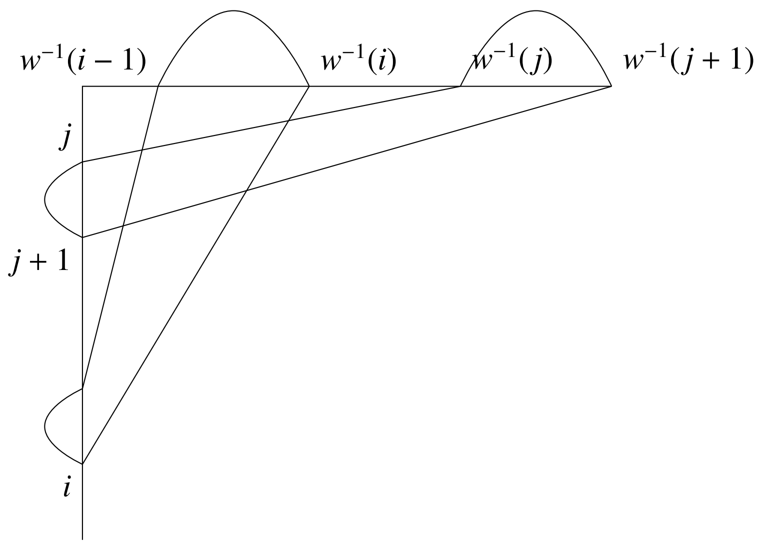

Figure 2.13: The proof of lemma 2.3.2 when i is even and j is odd.

$w^{-1}(i-1)$. This is illustrated in figure 2.3.

Case j is odd and i is odd: The equations for $M \in \mathfrak{sp}_n$ require $M_{i,j} = M_{j+1,i+1}$. In order for w to meet the definition of a pair permutation and for $w^{-1}(i) < w^{-1}(j)$, ι restricted to $i, i+1, j, j+1$ must form a rainbow involution hence $w^{-1}(i) < w^{-1}(j) < w^{-1}(j+1) < w^{-1}(i+1)$. This is illustrated in figure 2.3.

Case i is even and j is even: The equations for $M \in \mathfrak{sp}_n$ require $M_{i,j} = M_{j-1,i-1}$. In order for w to meet the definition of a pair permutation and for $w^{-1}(i) < w^{-1}(j)$, ι restricted to $i-1, i, j-1, j$ must form a rainbow involution hence $w^{-1}(j-1) < w^{-1}(i-1) < w^{-1}(i) < w^{-1}(j+1)$. This is illustrated in figure 2.3. □

Lemma 2.3.3 $\dim Y_\iota = (2n)^2 - \dim X^w$

Proof Notice that $\dim Y_\iota = (8n^2 + n - l(\iota))/2 = (8n^2 + n - (n + 2c + 4r))/2$ and that $\dim X^w = l(w) = c + 2r$, where c is the number of countryside involutions

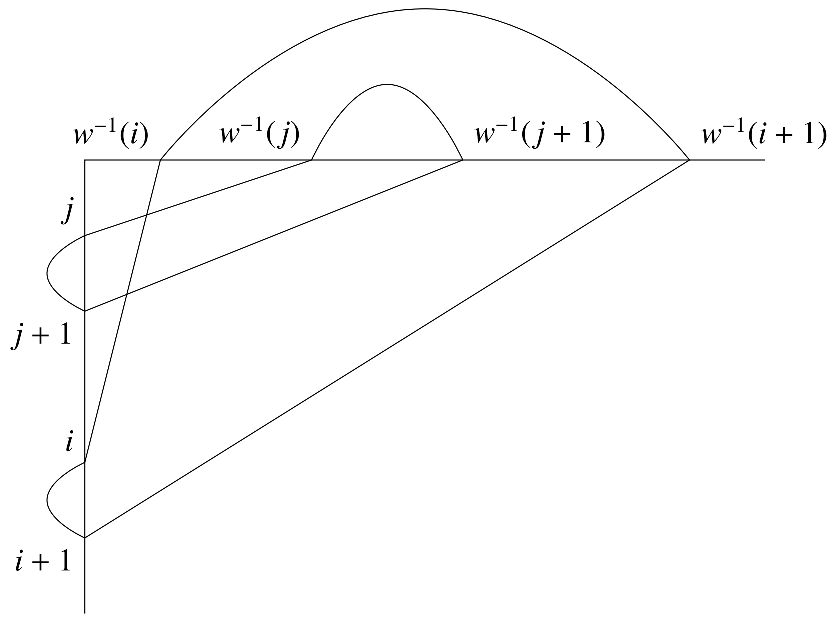

Figure 2.14: The proof of lemma 2.3.2 when i is odd and j is odd.

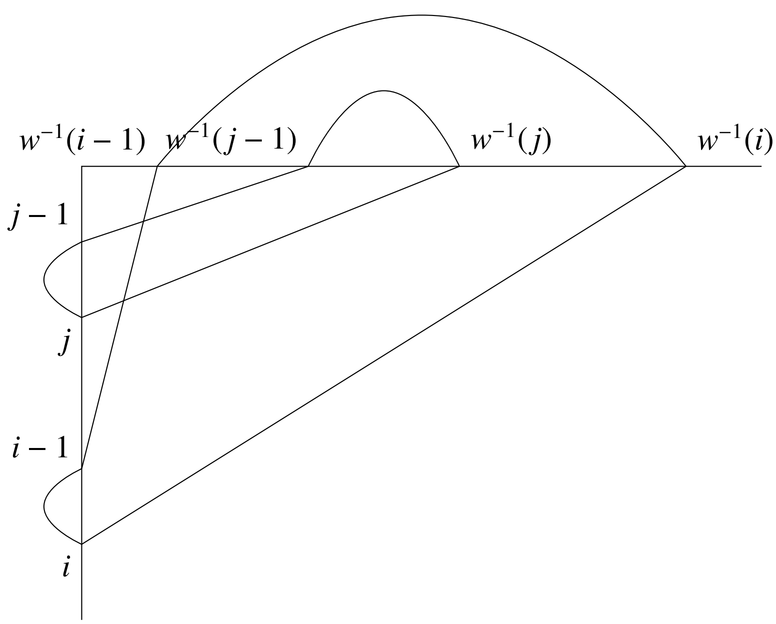

Figure 2.15: The proof of lemma 2.3.2 when i is odd and j is odd.

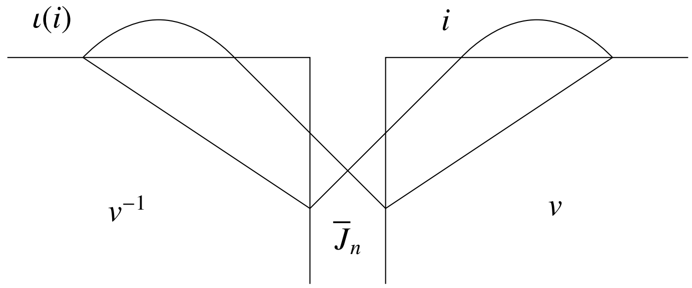

Figure 2.16: The proof of lemma 2.16.

contained in ι and r is the number of rainbow involutions contained in ι. Then we can observe that $\dim X^w = (2n)^2 - \dim Y_\iota$. □

Lemma 2.3.4 *The elements $v \in S_{2n}$ of minimal length such that $w\overline{J}w^{-1} = \iota$ are the pair permutations for ι.*

Proof The pair permutations are of minimal length because we minimize the number of wire crossings. We will therefore need to show that $wJw^{-1} = \iota$, or equivalently, that $w^{-1}Jw = \iota$. Let $w(i)$ be odd, then $Jw(i) = w(i) + 1$ and $w^{-1}(w(i)+1) = \iota(i)$. Similarly, if $w(i)$ is even, $Jw(i) = w(i) - 1$ and $w^{-1}(w(i)-1) = \iota(i)$. The best way to see this is to follow a pair of strands used to make a circle in the diagram description of pair permutations. This idea is most easily seen in figure 2.16. □

Proof of theorem 2.3.1 By [Bri03] the Y_ι degenerate to a union of Schubert varieties and it is enough to show that $(B_-\backslash Y_\iota) \cap (B_-\backslash\overline{B_-wB_-}) \neq \varnothing$, where $X^w = B_-\backslash B_-wB_+$. By Borel's fixed point theorem, $(B_-\backslash Y_\iota) \cap (B_-\backslash\overline{B_-wB_-}) \neq \varnothing$ if and only if $(B_-\backslash Y_\iota \cap X^w)^{T_{Sp_n}} \neq \varnothing$. But

$$\begin{aligned}
(B_-\backslash Y_\iota \cap X^w)^{T_{Sp_n}} &= (B_-\backslash Y_\iota)^{T_{Sp_n}} \cap (X^w)^T \\
&= (B_-\backslash Y_\iota)^{T_{Sp_n}} \cap [1, w] \\
&= \cup_{\iota' \leq \iota} (B_-\backslash Y^\circ_{\iota'})^{T_{Sp_n}} \cap [1, w] \\
&= \{v : vJV^{-1} = \iota\}
\end{aligned}$$

where v is of minimum length. Lemmas 2.3.2, 2.3.3 and 2.3.4 now complete the proof. □

CHAPTER 3

UNIONS OF MATRIX SCHUBERT VARIETIES

We compute a Gröbner basis for the ideal defining a union of schemes given by northwest rank conditions with respect to the "antidiagonal term order." By this we mean any scheme whose defining equations are of the form "all $k \times k$ minors in the northwest $i \times j$ submatrix of a matrix of variables," where i, j, and k can be filled in with varying values. On the algebraic side, this chapter provides access to a larger set of examples of determinantal varieties. On the geometric side, we have computed a generating set for the ideal defining a union of matrix Schubert varieties, including the unions described in 2.3.1.

3.1 Preliminaries

Lemma 3.1.1 *If $J \subseteq K$ are homogeneous ideals in a polynomial ring such that init J = init K then $J = K$.*

Proof The standard monomials (those monomials not in the initial ideal) are the same for init I and init J and give bases of both R/J and R/K, so the surjection $R/J \twoheadrightarrow R/K$ is also an injection. $\square$

Lemma 3.1.2 *Let I_A and I_B be ideals that define varieties of northwest rank conditions and let $g_A \in I_A$ and $g_B \in I_B$ be determinants with antidiagonals A and B respectively such that $A \cup B$ is the antidiagonal of a submatrix M. Then* $\det(M)$ *is in $I_A \cap I_B$.*

Proof Let $V_M = V(\det(M))$, $V_A = V(I_A)$ and $V_B = V(I_B)$, it is enough to show that $V_A \subseteq V_M$ and $V_B \subseteq V_M$ and, by relabeling just that $V_A \subseteq V_B$.

Let n be the number of rows (also the number of columns) in the submatrix M. Assume that the antidiagonal A is of length $r+1$, with left-most dot in column $t+1$ (hence in row $n-t$) of M and right-most dot in column c. Notice $c \geq (t+1)+(r+1)$, with equality if A occupies a contiguous set of columns, so matrices in V_A have rank at most r in the northwest $(n-t)\times(t+r+2)$ and hence have rank at most $r+(n-t-r-2) = n-t-2$ in the northwest $(n-t)\times n$, as we can add at most one to the rank in each additional column. Further, by the same principle, moving down t rows, the northwest $n\times n$, i.e. the whole matrix, has rank at most $n-t-2+t = n-2$, hence has rank at most $n-1$ and so is in V_M. □

In figure 3.1, the elements of the antidiagonal of g_A in the proof are shown with filled dots, while the entries of the antidiagonal of M that are only in the antidiagonal of g_B are shown with unfilled dots. The rank conditions $r+n-c$ and $r+n-c+t$ are those implied by the rank condition r met by all matrices in V_A.

3.2 Formula for the Generators of $\cap I_i$

Fix a set of antidiagonals $A_1, \ldots, A_n$ such that A_i is the antidiagonal of a Fulton generator of I_{π_i}. The generator $g_{A_1,\ldots,A_n}$ is given by

$$g_{A_1,\ldots,A_n} = \sum (-1)^{\mathrm{sign}(f)} \prod_{b\in\cup A_i} m_{\mathrm{row}(b),f(b)}$$

where the sum is over all possible functions

$$f : \{\cup A_i\} \to \mathrm{columns}(\cup A_i)$$

subject to these restrictions:

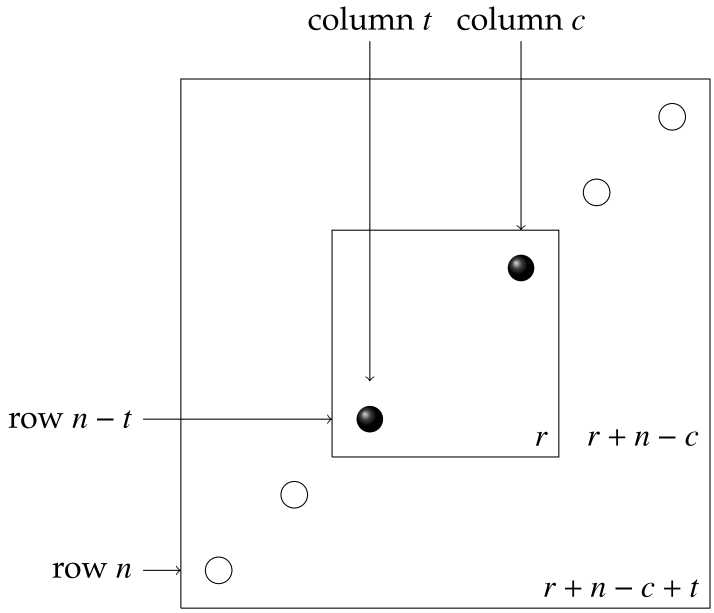

Figure 3.1: The proof of lemma 2.16.

1. For each column c, $|f^{-1}(c)| = |c \cap (\cup A_i)|$. Equivalently, $g_{A_1,\dots,A_n}$ is a weight vector under the T-action on the right.
2. f is injective on each A_i.
3. For each box $b \in A_i$, $f(b) \leq \max\{c : c \text{ is a column of a set } S \text{ containing } A_i\}$, where S is any collection of boxes no two boxes in the same row such that if $\text{row}(a) \geq \text{row}(b)$ then $\text{column}(a) \leq \text{column}(b)$.

Here sign($\bullet$) is an extension of the notion of the sign of a permutation: order the boxes containing dots first by row and then by column, working west to east, and similarly order the occupied columns from left to right. Use this ordering to make a partial permutation corresponding to the function $\bullet$. The sign of this partial permutation (the parity of the number of boxes in its diagram) will be sign($\bullet$).

Theorem 3.2.1 *The intersection of the I_{π_i} is generated by the $g_{A_1 \ldots A_n}$:*

$$\langle g_{A_1 \ldots A_n} : A_i \text{is an antidiagonal of a Fulton generator of } I_{\pi_i} \rangle = \bigcap_{i=1}^{n} I_{\pi_i}$$

The proof will come after some examples.

3.3 Examples

The generator for the three antidiagonals shown in figure 3.2 would be

$$\begin{vmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{vmatrix} \begin{vmatrix} m_{1,2} & m_{1,3} & m_{1,3} \\ m_{2,2} & m_{2,3} & m_{2,3} \\ m_{3,2} & m_{3,3} & m_{3,4} \end{vmatrix} \begin{vmatrix} m_{4,2} & m_{4,3} \\ m_{5,2} & m_{2,2} \end{vmatrix}.$$

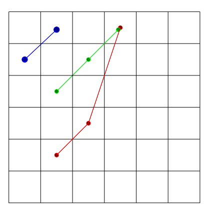

Figure 3.2: Generator example

The generator for the three antidiagonals shown in figure 3.3 would be

$$\begin{vmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{vmatrix} \begin{vmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{vmatrix}.$$

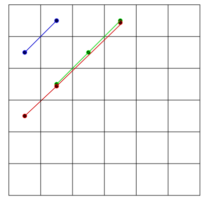

Figure 3.3: Generator example

The generator for the three antidiagonals shown in 3.4 would be

$$\begin{vmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{vmatrix} \left(\begin{vmatrix} m_{2,1} & m_{2,2} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} \end{vmatrix} \begin{vmatrix} m_{1,1} & m_{1,5} \\ m_{5,1} & m_{2,2} \end{vmatrix} - \begin{vmatrix} m_{1,1} & m_{1,2} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,4} \end{vmatrix} \begin{vmatrix} m_{2,1} & m_{1,5} \\ m_{5,1} & m_{5,5} \end{vmatrix} \right.$$

$$\left. + \begin{vmatrix} m_{1,1} & m_{1,2} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,4} \end{vmatrix} \begin{vmatrix} m_{3,1} & m_{2,5} \\ m_{5,1} & m_{5,5} \end{vmatrix} - \begin{vmatrix} m_{1,1} & m_{1,2} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{4,4} \end{vmatrix} \begin{vmatrix} m_{4,1} & m_{3,5} \\ m_{5,1} & m_{5,5} \end{vmatrix} \right)$$

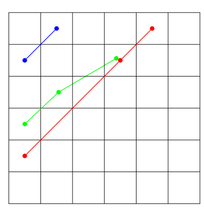

Figure 3.4: Generator example

Proof of Theorem 3.2.1

Theorem 3.2.1 follows from the proceeding list of lemmas and the preceeding more general lemmas. Most importantly is the key:

Lemma 3.3.1 $g_{A_1,\ldots,A_n} \in I_i$ *for each i.*

Proof Fix i, we will show that $g_{A_1,\ldots,A_n} \in I_i$. Let S be the antidiagonal containing A_i which contains the column c and achieves the maximum from condition 3. That is, S is a collection of boxes containing A_i such that if $\text{row}(a) \geq \text{row}(b)$ then $\text{column}(a) \leq \text{column}(b)$ and S contains at most 1 box per row. Notice that $|S| \geq |A_i|$. Also, note that the determinant with antidiagonal term S is in I_i by a previous lemma. For each f from the algorithm, set $S' = f(S)$. By condition 3 $\max\{c' : c' \text{ is a column of } S'\} = c$. Then,

$$\begin{aligned} g_{A_1,\ldots,A_n} &= \sum (-1)^{\text{sign}(f)} \prod_{b \in \cup A_i} m_{\text{row}(b),f(b)} \\ &= \sum_{S'} \left(\sum_{f\, s.t.\, f(S)=S'} (-1)^{\text{sign}(f)} \prod_{b \in S} m_{\text{row}(b),f(b)} \right) \\ &\quad \left(\sum_{f'\, s.t.\, f'(S)=S'} (-1)^{\text{sign}(f')} \prod_{b \in S^c} m_{\text{row}(b),f(b)} \right) \end{aligned}$$

The left factor in each summand is a determinant of the rows of S and the columns given by S' which are, by construction, at most as large as the columns in S and hence in any ideal generated by northwest rank conditions and containing a determinant with antidiagonal S. □

Therefore, $I \subseteq \cap I_{\pi_i}$. Note that there are many such $g_{A_1,\ldots,A_n}$, generically $\prod_{i=1}^{n}(\text{number of generators of } I_{\pi_i})$ many.

Lemma 3.3.2 *init* $g_{A_1,\ldots,A_n} =$ *the union of the antidiagonals* $A_1 \cup \cdots \cup A_n$.

Proof Let ι be the function that sends every dot to its original location. We claim that init $g_{A_1,\ldots,A_n} = (-1)^{\text{sign}(i)} \prod m_{r,i(r,v)}$. We show this by induction on the number of places where an arbitrary function f meeting the requirements above differs from ι. If f differs from ι in exactly 2 places then after canceling variables that are the same we are left with a 2×2 determinant. ι corresponds to the antidiagonal. Assume the result is true for any f that differs from ι in $2m$ places. Then, by the same argument as above, an f that differs in an additional 2 places provides an even smaller term, so the original ι is the largest term. □

Lemma 3.3.3 *The generating set $\{g_{A_1,\ldots,A_n}\}$ given above is a Gröbner basis for I.*

Proof That $\operatorname{init} I = \langle \operatorname{init} g_{A_1,\ldots,A_n} \rangle$ where the $g_{A_1,\ldots,A_n}$s are the generators given by the above plan follows from lemma 3.3.2. □

Note that we have not used any facts about the I_{π_i} besides that they are ideals corresponding to (reduced, but not necessarily irreducible) schemes that are determined by northwest rank conditions. For example, the ideal

$$\langle m_{1,1}, \begin{vmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{vmatrix} \rangle$$

is such a "matrix Schubert scheme," however, any such scheme

$$I = \cap_{I_\pi \geq I} I_\pi$$

in any case.

CHAPTER 4

EQUATIONS OF ORBIT CLOSURES

4.1 Sufficient Steps to Get Equations of Orbit Closures

While we did not succeed in finding equations of orbit closures of Sp_n, we have made some progress in this direction. The description of the basic elements of the poset of orbit closures contained in the set $\mathcal{A}$ (see proposition 2.2.5) reduces the number of orbit closures whose equations we must find to those in $\mathcal{A}$ because the intersections are reduced. Since the Y_ι degenerate to compatibly Frobenius split subvarieties they are compatibly split. Therefore

$$\text{init } I(Y_\iota) = \text{init } \cap_{\iota' \in \mathcal{A}, \iota' \geq \iota} I(Y_{\iota'}) = \cap_{\iota' \in \mathcal{A}, \iota' \geq \iota} \text{init } I(Y_{\iota'})$$

by [Knu]. The ideal for each of the init $I(Y_{\iota'})$ is given by a combination of theorems 2.3.1 and 3.2.1. Then, we are given a suggestion as to those equations by their degenerations into a union of Schubert varieties and the equations for unions of matrix Schubert varieties given in theorem 3.2.1. For each Gröbner basis element of $I(\text{init } Y_\iota)$, we must find an equation which holds on Y_ι and which has initial form the Gröbner basis element with which we began. This will then be a Gröbner basis of Y_ι.

4.2 Known Equations of Orbit Closures

The **pfaffian** of an antisymmetric matrix is a choice of the square root of its determinant, which is a square. Since we shall use it as a generator for an ideal we shall not concern ourselves with which choice of the square root.

Proposition 4.2.1 *Assume ι has one symplectically essential box at $(2r-1, 2r)$ and that that box has associated rank condition $2r-2$. Then the reduced variety Y_ι is defined by the pfaffians of rows and columns $\{1, \ldots, 2r\}$ of MJM^T where M is a $2n \times 2n$ matrix of variables.*

Proof Expand the symplectic diagram to the full Rothe diagram using antisymmetry of the matrices and apply Fulton's theorem. Then recall that the determinant of an antisymmetric matrix, which is the only Fulton generator in this case, is a square with square root the pfaffian of that matrix. □

Lemma 4.2.2 *Providing the equations for the reduced varieties associated to the Y_ι requires calculating the radical of a variety with ideal given by determinants of MJM^T.*

Proof We wish to describe the set that corresponds to the orbit Y_ι. We do this by applying the map given by [RS90] $M \mapsto MJM^T$ to a matrix of variables M. Then, we apply Fulton's Theorem (theorem 1.2.1) and take the radical to get the correct description for the reduced scheme. □

This has proven computationally infeasible even in fairly small examples. We have had some success with computing a few examples using [GS], summarized in table 4.1. In table 4.1 $\det((MJM^T)_{R,C})$ is the determinant of rows R and columns C of MJM^T, where M is a matrix of variables. Similarly, $\mathrm{pf}((MJM^T)_{R,C})$ is the pfaffians of rows R and columns C of MJM^T, where M is a square matrix of variables of size the number of elements that ι is permuting. Note that the equations found in table 4.1 are mostly not a Gröbner basis for the ideals they generate.

Table 4.1: Generators for some $I(Y_\iota)$

ι	Ideal Generators
$4321 \oplus \overline{J}_{n-2}$	$\mathrm{pf}((MJM^T)_{(1,2),(1,2)})$, $\mathrm{pf}((MJM^T)_{(1,3),(1,3)})$
216543	$\mathrm{pf}((MJM^T)_{(1,2,3,4),(1,2,3,4)})$, $\mathrm{pf}((MJM^T)_{(1,2,3,5),(1,2,3,5)})$
351624	$\mathrm{pf}((MJM^T)_{(1,2),(1,2)})$, $\mathrm{pf}((MJM^T)_{(1,2,3,4),(1,2,3,4)})$

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